Exercise 61

Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

$$y = x^4 - x^6$$

Solution

To find the y-intercept, plug in $x = 0$ to the function.

$$y = (0)^4 - (0)^6 = 0$$

Therefore, the y-intercept is $(0, 0)$. To find the x-intercept(s), set $y = 0$ and solve the equation for x.

$$x^4 - x^6 = 0$$

$$x^4(1 - x^2) = 0$$

$$x^4(1 + x)(1 - x) = 0$$

$$x = 0 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 1$$

Therefore, the x-intercepts are $(0, 0)$ and $(-1, 0)$ and $(1, 0)$. Calculate the limit of the function as $x \to \pm\infty$. In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} (x^4 - x^6) = \lim_{x\to\infty} x^6 \left(\frac{1}{x^2} - 1\right) = \lim_{x\to\infty} \frac{\frac{1}{x^2} - 1}{\frac{1}{x^6}} = \frac{0 - 1}{0} = -\infty$$

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{u\to\infty} [(-u)^4 - (-u)^6] \\
&= \lim_{u\to\infty} (u^4 - u^6) \\
&= \lim_{u\to\infty} u^6 \left(\frac{1}{u^2} - 1\right) \\
&= \lim_{u\to\infty} \frac{\frac{1}{u^2} - 1}{\frac{1}{u^6}} \\
&= \frac{0 - 1}{0} \\
&= -\infty
\end{aligned}$$

Below is a graph of the function versus x.